Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE ANALYST.

Vol. I. June, 1874. No. 6.

THE POTENTIAL FUNCTION.

BY PROF. D. TROWBRIDGE, WATERBURGH, N. Y.

1. There is a function of constant occurrence in all investigations that relate to attractions, and it is of so much importance that I purpose to give, in this paper, a brief explanation of its nature, composition and use. Several terms have been proposed to express its nature, and by which it may be known. It has been proposed to call it the *Integral Proximity* of the attracting mass; Mr. Green and Prof. MacCullagh called it the *Potential* of the attracting mass; and Dr. Whewell proposed to call it the *Integral Potentiality*, or *Integral Attractivity* (*Hist. Inductive Sciences*, Vol. III, p. 76). The term by which it is now generally known, however, is *Potential*. It would seem that the term was chosen because the function has some relation to the *power* of the body, while the actual force exerted by the mass results at once from the function, by differentiation. The potential function is often called V.

2. If m be the mass of a particle, and r its distance from an attracted point, then, according to Newton's law of gravity, the force of attraction of the mass m, at the distance r and along the line r, is

$$\frac{m}{r^2}. \quad \text{If we put } V = \frac{m}{r}, \text{ then} -D_r V = \frac{m}{r^2}.$$

If m be the differential of a mass M, then V will be the sum of all the particles of M, divided by their distances from the attracted point. The integral calculus will enables us to find the value of this sum. Since $m = d\,M$, we shall have

$$V = \int \frac{d\,M}{r} \quad \text{.......................} \quad (1)$$

the integral extending over the whole attracting mass.

Let x, y, z be the rectangular coördinates of any point of the body M,

a, b, c the rectangular coördinates of the attracted point; then if ρ be the density of M at the point $x\ y\ z$, we shall have

$$d\,M = \rho\, dx\, d\,y\, d\,z,\ r^2 = (a-x)^2 + (b-y)^2 + (c-z)^2 \ldots\ldots (2)$$

and

$$V = \iiint \frac{\rho\, d\,x\, d\,y\, d\,z}{\sqrt{(a-x)^2 + (b-y)^2 + (c-z)^2}}, \ldots\ldots\ldots\ldots (3)$$

the limits of x, y and z being derived from the equation of the surface of the attracting body. In equation (3) a, b, c are entirely independent of ρ, x, y, z, and if we let X, Y and Z be the force of attraction resolved along the coördinate axes x, y, z respectively, we shall have

$$-D_a V = X = -\iiint \frac{\rho\,(a-x)d\,x\,d\,y\,d\,z}{[(a-x)^3 + (b-y)^2 + (c-z)^2]^{\frac{3}{2}}},$$

$$-D_b V = Y = -\,-\iiint \frac{\rho\,(b-y)d\,x\,d\,y\,d\,z}{[(a-x)^2 + (b-y)^2 + (c-z)^2)]^{\frac{3}{2}}},$$

$$-D_c V = Z = -\iiint \frac{\rho\,(c-z)d\,x\,d\,y\,d\,z}{[(a-x)^2 + (b-y)^2 + (c-z)^2]^{\frac{3}{2}}} \ldots\ldots (4)$$

From these equations we easily find

$$\int (D_a V d\,a + D_b V d\,b + D_c V d\,c) = V = -\int (X d\,a + Y d\,b + Z\,d\,c) + C. (5)$$

The reader will understand that $D_a V$ is a partial differential coefficient of V with respect to a, on the supposition that V and a are the only varible quantities in equation (3). The same is true of $D_b V$ and $D_c V$. The subject of partial differential coefficients should be thoroughly understood by the student. In some cases the variation of the quantities is entirely arbitrary, as in equations (4), and it is made purely for analytical convenience; and in others the variation is partial, but *may* exist independent of other quantities.

3. The potential must be calculated for the point which we suppose attracted by the mass whose potential we calculate. Thus, we may suppose the potential calculated for *any* point $a\ b\ c$, *external* to the attracting mass, or for *any* point $a\ b\ c$, *internal* to the attracting mass, but the calculation will be different for the two cases.

Except for a few cases the calculation of the potential seems to be one of extreme difficulty, and even beyond the present resources of analysis, if the complete expression be required. Laplace devised a method, however, for resolving the integral, which gives the potential, into a series. The terms of this series possess some very important properties which are very useful in the higher parts of mathematical physics. Certain coefficients in this series are technically called *Laplace's Coefficients.* It is not my purpose to discuss, in this paper, the properties of these coefficients. The discussion is usually thought to be very difficult; and as it is usually presented, it is, but it is not necessarily so.

4. Now suppose the value of V found; it will be a function of the coördinates a, b, c. Suppose we wish to transform the coördinates from a, b, c to a', b', c'. Let α, β, γ, α', β', γ', and α'', β'', γ'', be the angles made by a, b, c with the axes of a', b', c', respectively. Then we shall have (see Davies' *Analytical Geometry*, Book IX,)

$$a = a' \cos \alpha + b' \cos \beta + c' \cos \gamma,\ b = a' \cos \alpha' + b' \cos \beta' + c' \cos \gamma',$$
$$c = a' \cos \alpha'' + b' \cos \beta'' + c' \cos \gamma'' \ldots\ldots\ldots\ldots\ldots (6)$$

Since we now suppose V to be a function of a', b', c', we shall have

$$D_a' V = D_a V D_a' a + D_b V D_a' b + D_c V D_a' c$$
$$= - [X D_a' a + Y D_a' b + Z D_a' c] \ldots\ldots\ldots\ldots (7)$$

From equation (6) we have

$$D_a' a = \cos \alpha \quad D_a' b = \cos \alpha',\ D_a' c = \cos \alpha'';$$

and these substituted in equation (7) give

$$D_a' V = - [X \cos \alpha + Y \cos \alpha' + Z \cos \alpha''] = - X',$$

the force in the direction of the axis of α' (as Analytical Geometry wil show). In a precisely similar manner we could find

$$D_b' V = - Y', \text{ and } D_c' V = - Z'.$$

We hence conclude that, if we have V calculated for *any* axes whatever we can find the components of the attractive force in the direction of any other axes, by substituting for the original co-ordinates their values in terms of the new ones, and then taking the partial differential coefficients with respect to the new coördinates.

5. If we substitute polar coördinates in equation (3), we shall have, if we make r, θ, ω the polar coördinates of the attracted point, and, r', θ', ω' the polar coördinates of any point of the solid,

$$a = r \sin \theta \cos \omega,\ b = r \sin \theta \sin \omega,\ c = r \cos \theta \ldots\ldots\ldots (8)$$
$$x = r' \sin \theta' \cos \omega',\ y = r \sin \theta' \cos \omega',\ z = r' \cos \theta' \ldots\ldots (9)$$
$$r^2 = a^2 + b^2 + c^2,\ r'^2 = x^2 + y^2 + z^2 \ldots\ldots\ldots\ldots (10)$$
$$(a-x)^2 + (b-y)^2 + (c-z)^2 = r^2 + r'^2 - 2rr'[\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\omega - \omega')]$$
$$d\,x\,d\,y\,d\,z = r'^2 \sin \theta'\, d\,r'\, d\,\theta'\, d\,\omega'.$$

With these values we have

$$V = \iiint \frac{\rho\, r'^2 \sin\theta'\, d\, r'\, d\, \theta'\, d\, \omega'}{[r^2 + r'^2 - 2\, rr'(\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\omega - \omega')]^{\frac{1}{2}}} . (11)$$

This equation is very much simplified if we place the origin of coördinates at the attracted point, for then $r = 0$, and equation (11) becomes

$$V = \iiint \rho\, r' \sin\theta'\, d\, r'\, d\, \theta'\, d\, \omega' \ldots\ldots\ldots\ldots (12)$$

From this equation we see that the value of V is continuous, or does not become infinite as appears possible from equation (11).

6. Thus far we have considered V a continuous function, and made up of the sum of all the particles of the attracting mass, divided by their respective distances from the attracted point. There is another case in which we may consider the attracting particles (or masses) as independent and finite in number. Thus, if we wish to find the potential of the masses m', m'', &c., for their attractive influence on the mass m, the distances being r', r'', &c., we should have

$$V = + \frac{m'}{r'}\ \frac{m''}{r''} + \frac{m'''}{r'''} + .. = \Sigma \left(\frac{m'}{r'}\right), \text{ suppose} \ldots\ldots (13)$$

When we wish to find the potential for the *disturbing force* exerted by m', m'', &c., on the motion of m, the form of the expression becomes changed, and it is usually called R, the *Disturbing Function*. This form of V is employed in the Lunar and the Planetary Theory.

7. We shall now show how to find V for a sphere of uniform density. If we use equation (12) we must integrate with respect to ω' between the limits 0 and $2\,\pi$; r', between the limits r'_1 and r'_2 for an external point; and between the limits $r_1{}'$ and 0 for an internal point. If we put $\mu = \cos\theta'$ we shall have

$$V = \pi\,\rho \int (r'^2_1 - r'^2_2) d\,\mu \ldots (14) \quad \text{and} \quad V = \pi\,\rho \int (r'^2_1\, d\,\mu \ldots\ldots (15)$$

The equation of the generating circle of the sphere is

$$x^2 + y^2 = a^2 \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Put c = the distance between the attracted point and the center of the sphere, and take this line for the axis of x. We then have

$$x = r' \cos\theta' - c,\ y = r' \sin\theta'.$$

These values in (16) give

$$r'^2 - 2\,c\,r' \cos\theta' = a^2 - c^2, \qquad r' = c\,\mu \pm \sqrt{a^2 - c^2 + c^2\,\mu^2}.$$

These two values of r' give

$$r_1' = c\,\mu + \sqrt{a^2 - c^2 + c^2\,\mu^2},\ r'_2 = c\,\mu - \sqrt{a^2 - c^2 + c^2\,\mu^2}.$$

$$r'^{\,2}_1 - r'^{\,2}_2 = 4\,c\,\mu\,\sqrt{a^2 - c^2 + c^2\,\mu^2}$$

$$r'^{\,2}_1 = 2\,c^2\,\mu^2 + a^2 - c^2 + 2\,c\,\mu\,\sqrt{a^2 - c^2 + c^2\,\mu^2}.$$

The limits of μ for an external point are $\mu = +1$, and $c^2\,\mu^2 = c^2 - a^2$; and for an internal point the limits of μ are $+1$ and -1. The equation (14) now gives, if $c^2\,\mu'^2 = c^2 - a^2$,

$$V = 4\,\pi\,\rho\int_{+\mu'}^{+1} c\,\mu\,d\,\mu\,\sqrt{a^2 - c^2 + c^2\,\mu^2} = \frac{4\,\pi\,\rho\,a^3}{3\,c} = \frac{\text{mass}}{c}.$$

Equation (15) gives

$$V = 2\,\pi\,\rho\int_{-1}^{+1} d\,\mu\,(2\,c^2\,\mu^2 + a^2 - c^2 + 2\,c\,\mu\,\sqrt{a^2 - c^2 + c^2\,\mu^2}) = 2\,\pi\,\rho\,a^2 - \tfrac{2}{3}\,\pi\,\rho\,c^2.$$

We can find the potential of a homogeneous spheroid of revolution, for a point situated in the axis of revolution, by a similar process, though the work is more complicated. Since the values of X, Y, Z can be calculated directly, we can find V for the spheroid by equation (5), the value of C being made the value of V for the center of the spheroid.

~~~~~~

SOLUTION OF TWO SIMILAR INDETERMINATE PROBLEMS.

BY GEORGE R. PERKINS, LL. D., UTICA, N. Y.

1. *Find three square numbers in arithmetical progression, such that if from each its root be subtracted, the remainders shall be squares.*

2. *Find three square numbers in arithmetical progression, such that if to each its root be added, the sums shall be squares.*

We will assume for the three square numbers as follows:

$$\left\{\tfrac{1}{4}(x + x^{-1}) \pm \tfrac{1}{2}\right\}^2;\ \left\{\tfrac{1}{4}(y + y^{-1}) \pm \tfrac{1}{2}\right\}^2;\ \left\{\tfrac{1}{4}(z + z^{-1}) \pm \tfrac{1}{2}\right\}^2.$$

The *upper* sign corresponds with the first problem, and the *lower* sign with the second problem.

Since the following condition is true for all values of x,

$$\left\{\tfrac{1}{4}(x + x^{-1}) \pm \tfrac{1}{2}\right\}^2 \mp \left\{\tfrac{1}{4}(x + x^{-1}) \pm \tfrac{1}{2}\right\} = \left\{\tfrac{1}{4}(x - x^{-1})\right\}^2, \ldots\ldots(1)$$

we see that the square numbers as above assumed, will satisfy the con-
~~~~~~